A tensor-based framework for studying eigenvector multicentrality in multilayer networks

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Contributed by H. Vincent Poor, May 21, 2019 (sent for review January 25, 2018; reviewed by Na Li, Muriel Medard, and Edmund Yeh)

Centrality is widely recognized as one of the most critical measures to provide insight into the structure and function of complex networks. While various centrality measures have been proposed for single-layer networks, a general framework for studying centrality in multilayer networks (i.e., multicentrality) is still lacking. In this study, a tensor-based framework is introduced to study eigenvector multicentrality, which enables the quantification of the impact of interlayer influence on multicentrality, providing a systematic way to describe how multicentrality propagates across different layers. This framework can leverage prior knowledge about the interplay among layers to better characterize multicentrality for varying scenarios. Two interesting cases are presented to illustrate how to model multilayer influence by choosing appropriate functions of interlayer influence and design algorithms to calculate eigenvector multicentrality. This framework is applied to analyze several empirical multilayer networks, and the results corroborate that it can quantify the influence among layers and multicentrality of nodes effectively.

multilayer networks | eigenvector centrality | PageRank centrality

Centrality quantifies the importance of nodes in a graph and has been widely studied to understand the structure and function of complex networks (1, 2). For example, it can be used to identify the most influential person in an online social network (3), the most crucial artery in transport congestion (4), or the most important financial institution in the global economy (5). Over 30 different centrality measures (e.g., degree centrality, betweenness centrality, closeness centrality, eigenvector centrality, and control centrality) have been examined in the literature (6–9). Among these, eigenvector centrality, defined as the leading eigenvector of the adjacency matrix of a graph, has received increasing attention (10, 11). It is worth noting that PageRank, a variant of eigenvector centrality, is the primary algorithm used in Google's search engine (12, 13).

Notably, most previous studies have focused on eigenvector centrality in a single-layer network, in which all nodes/links are assumed to be of the same type (centrality homogeneous). As revealed recently (14–19), many practical complex systems, ranging from the Internet to airline networks, have multiple types of nodes and/or links between nodes. Multilayer networks, which consist of multiple layers of nodes with intra- and interlayer links, can be used to model such complex systems. Fig. 1 shows 2 examples of multilayer networks (see *SI Appendix*, Fig. S1 for more examples). Simply aggregating a multilayer network into a single-layer one would obviously lead to a miscalculation of centrality. Recent work on eigenvector-like centrality in multilayer networks either assigned constant weights to predetermine interlayer influence [which can be regarded as the gain or loss of the interplay strength between 2 layers (20)] or focused on a special case of multilayer networks, i.e., the so-called multiplex networks (where all layers share the same set of nodes, and interlayer links exist only between counterpart nodes) (21–24). It is of significant interest to develop a framework for studying eigenvector-like centrality in general multilayer networks, hereafter referred to as eigenvector multicentrality. In this study, we introduce a tensor-based framework that enables the quantification of the relationship between interlayer influence and eigenvector multicentrality. It is challenging to compute eigenvector multicentrality of nodes in such a framework since interlayer influence and eigenvector multicentrality are interdependent. We prove the existence and uniqueness of eigenvector multicentrality for given appropriate forms of interlayer influence. We also design efficient algorithms to calculate it for 2 interesting scenarios. This framework offers an approach for modeling and quantifying the interlayer interactions in multilayer networks, providing a systematic way of characterizing eigenvector multicentrality. Experimental results based on several real-world multilayer networks corroborate our analytical results.

Results

A Tensor-Based Framework for Studying Eigenvector Multicentrality. In the calculation of eigenvector centrality of nodes in a single-layer network, a directed link to a node can be viewed as a

Significance

It is of significant interest to understand the structure and function of multilayer networks, which model many practical complex systems. Centrality, quantifying the importance of nodes in a graph, is widely recognized as one of the most effective measures. Nevertheless, a general framework for characterizing centrality in multilayer networks is still lacking. In this article, we fill this gap by developing a tensor-based framework for characterizing eigenvector multicentrality in general multilayer networks. We prove the existence and uniqueness of eigenvector multicentrality for 2 interesting scenarios, using the proposed framework. The results from empirical networks demonstrate that this framework helps us obtain a clear understanding of the eigenvector multicentrality of nodes.

Author contributions: S.H., J.C., J.Z., and H.V.P. designed research; M.W. performed research; M.W. and S.H. contributed new reagents/analytic tools; M.W., S.H., Y.Z., J.C., Y.S., Y.-Y.L., J.Z., and H.V.P. analyzed data; and M.W., S.H., J.C., Y.S., Y.-Y.L., J.Z., and H.V.P. wrote the paper.

Reviewers: N.L., Harvard University; M.M., Massachusetts Institute of Technology; and E.Y., Northeastern University.

Conflict of interest statement: Na Li received an honorarium from Zhejiang University for a seminar hosted by J.C.

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This article contains supporting information online at www.pnas.org/lookup/suppl/doi:10.1073/pnas.1801378116/-/DCSupplemental.

Published online July 17, 2019.

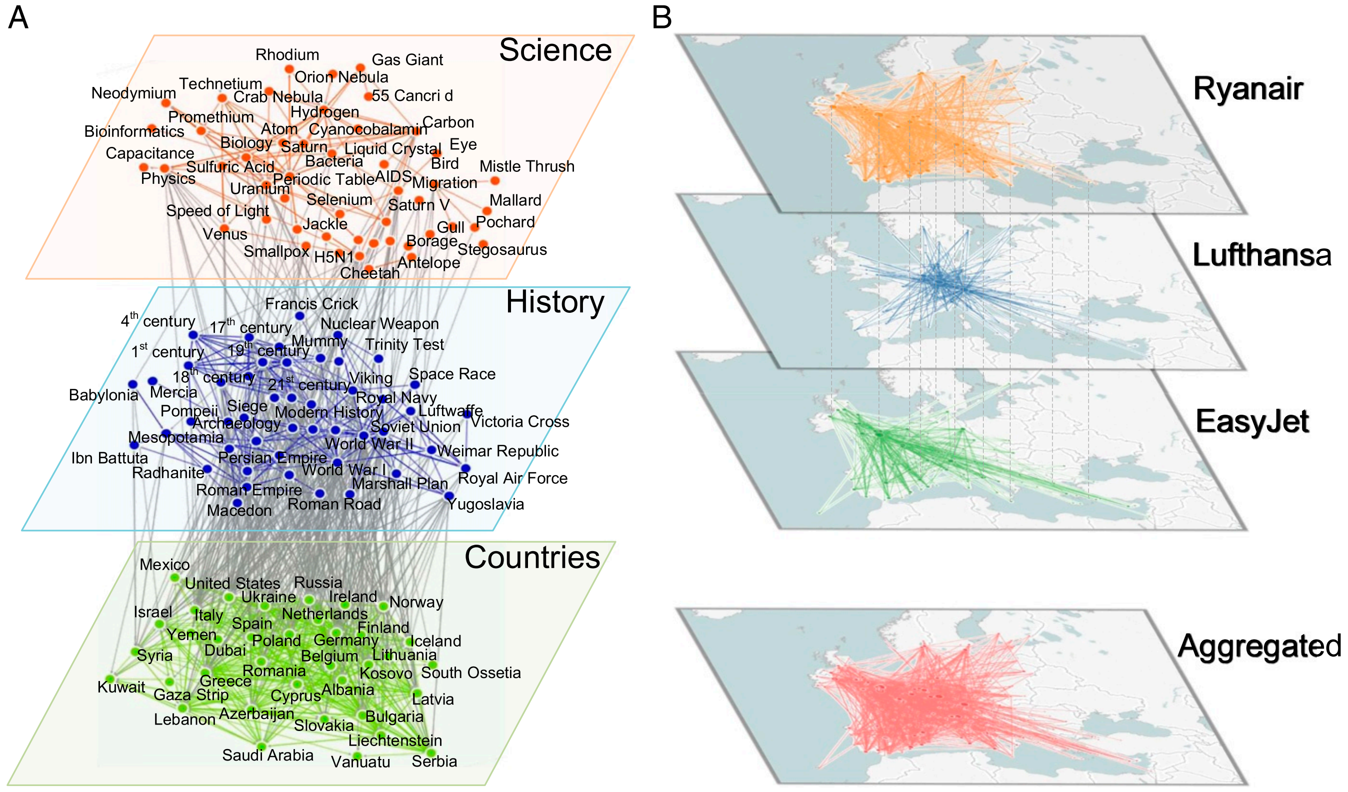

Fig. 1. Examples of multilayer networks. (*A*) A network of web pages in Wikipedia can be considered as a multilayer network. Layers represent subjects, and nodes denote words (or terms) connected by hyperlinks. Colorized links are intralayer links while gray ones are interlayer links. (*B*) A European airline network with 3 layers can be modeled as a multiplex network, which contains the same set of nodes in all layers. Intralayer links in 1 layer represent flight routes operated by an airline and interlayer links exist only between the same nodes (airports) in different layers. Only a portion of interlayer links is shown to not complicate the figure. A single-layer network obtained by simply aggregating all airports and flight routes is also shown at the bottom. The geographic data are provided by OpenStreetMap.

vote of support. Each node fairly propagates its entire centrality score to its neighbors recursively. The eigenvector centrality of a node is defined as the scores that it gathers from its neighbors after appropriate normalization in the steady state. Formally, the vector consisting of the eigenvector centrality of all nodes is defined as the leading left eigenvector of the adjacency matrix associated with the single-layer network. We generalize this definition to multilayer networks by taking into account interlayer influence among layers. Specifically, a multilayer network is modeled as $\mathcal{M} = (\mathcal{L}, \mathcal{E})$, where $\mathcal{L} = \{L_\alpha; \alpha = 1, 2, \cdots, K\}$ is a collection of graphs $L_\alpha = (V_\alpha, E_\alpha)$ representing layers in $\mathcal{M}$; $V_\alpha = \{v_{1,\alpha}, v_{2,\alpha}, \cdots v_{n_\alpha,\alpha}\}$ is the set of nodes, where n_α denotes the number of nodes, and E_α is the set of intralayer links in layer α; and $\mathcal{E} = \{E_{\alpha\beta} \subseteq V_\alpha \times V_\beta; \alpha, \beta = 1, 2, \cdots, K (\alpha \neq \beta)\}$ contains the interlayer links in $\mathcal{M}$ (16). To avoid confusion, we use Latin letters $\{i, j, \cdots\}$ to indicate nodes and Greek letters $\{\alpha, \beta, \cdots\}$ to indicate layers. Tensors provide a general mathematical tool to describe high-dimensional objects (25, 26), and notably, tensors have been used to study multilayer networks (27–29). Specifically, a fourth-order tensor $M^{i\alpha}_{j\beta}$, called the adjacency tensor, is used to encode a directed, weighted link from node i in layer α to node j in layer β (see *SI Appendix*, section I for further details on tensorial representations). We further introduce the influence tensor W^α_β, which is a second-order tensor measuring the interlayer influence from layer α to layer β. In our framework, the influence tensor W may be treated as a constant tensor when quantitative knowledge is available. The interaction tensor is defined as $H^{i\alpha}_{j\beta} = W^\alpha_\beta M^{i\alpha}_{j\beta}$, encoding the interaction from node i in layer α to node j in layer β. Intuitively, when the influence from layer α to layer β is greater than 1, the centrality scores propagating along the links from layer α to layer β will be magnified, and vice versa.

The second-order tensor $\Phi_{i\alpha}$ is defined as the solution to the tensorial equation

$$H^{i\alpha}_{j\beta} \Phi_{i\alpha} = \lambda_\beta \Phi_{j\beta}, \quad [1]$$

where λ_β is a coefficient related to layer β, and the Einstein notation (30) is adopted here (see *SI Appendix*, section I for further details). Because Φ is an eigenvector-like centrality, it is hereafter referred to as eigenvector multicentrality, and $\Phi_{i\alpha}$ represents the eigenvector multicentrality score of node i in layer α. In the calculation of eigenvector multicentrality, after each node propagates its entire multicentrality score to neighbors, the scores from layer α to layer β are multiplied by the influence coefficient W^α_β. Hence, we can obtain the eigenvector multicentrality via appropriate normalization in the steady state. Note that the normalizing coefficient λ_β could be different for different layers in a multilayer network. This differs from the eigenvector centrality in a single-layer network, where all nodes share a common normalizing coefficient λ_1 (namely, the leading eigenvalue of the adjacency matrix).

Many existing models can be incorporated into our eigenvector multicentrality framework by choosing the influence tensor W appropriately. For instance, in an author–document heterogeneous network, unweighted interlayer links connect documents to their authors. A directed unweighted intralayer link exists between 2 documents if 1 document refers to the other and the undirected weighted intralayer link between 2 authors represents their social tie. The multicentrality in such an author–document network was defined as the leading eigenvector of a stochastic matrix, which encodes the probabilities that a random surfer moves along intra- and interlayer links in a combined random walk process (22). Clearly, this model can be incorporated into our framework by setting the intra- and interlayer influence weights to predetermined constants. Further, a definition of multicentrality in multiplex networks has been proposed in ref. 20, considering the influence from counterpart nodes in other layers by importing the influence matrix $Q=(q_{\alpha\beta})\in\mathbb{R}^{K\times K}$. The influence matrix is nonnegative, and $q_{\alpha\beta}$ measures the influence of layer β on layer α. One can calculate eigenvector-like multicentrality of a multiplex network once Q has been obtained. Observe that in ref. 20 the influence matrix Q is predetermined and given as in a lower-dimensional form (i.e., a matrix), whereas in our framework the influence tensor W depends on the eigenvector multicentrality and hence they are interdependent. Moreover, interconnected multilayer networks have been proposed to predict diffusive and congestion processes (23), where the undirected unweighted interlayer links connect nodes to their counterparts in other layers. Eigenvector centrality in these networks is a special case of our framework where the interlayer influence is equal to 1.

Leveraging Prior Knowledge about Interlayer Interactions. Quantifying the influence tensor W is important to calculate multicentrality in our framework. As expected, the influence tensor W is typically a function of the adjacency tensor M and the multicentrality Φ, rather than being a constant. In practice, precisely predetermining the influence tensor is often infeasible. One advantage of our general framework lies in the leveraging of prior knowledge about the influence tensor W in diverse applications and the calculation of multicentrality even when W and Φ are interdependent.

Consider a typical scenario in which all nodes are centrality homogeneous and the layers are heterogeneous in a multilayer network. In such a scenario, the multicentrality scores of all nodes are comparable and can be represented by a vector $C\in\mathbb{R}^N$, where N is the number of all nodes. We call the eigenvector multicentrality in such a scenario global multicentrality. We normalize the multicentrality such that the vector C is defined over an N-dimensional simplex. For example, in a multilayer network consisting of web pages on different subjects (see *Multicentrality in Empirical Networks* for further details), we may need to compare the multicentrality scores of 2 web pages on different subjects. Clearly, the multicentrality score propagates differently along interlayer links and along intralayer links, owing to differences in the popularity of different subjects. Here, we assume that the importance of layer α is a function of the multicentrality of all nodes in layer α, denoted by $f(\Phi_{:\alpha})$, where the colon ":" indicates all elements of a given dimension (31). Notably, the function $f(\cdot)$ describing the layer importance is dependent on applications. The function f could be, for example, the L^1-norm $f(\Phi_{:\alpha})=\|\Phi_{:\alpha}\|_1$, which means the aggregated multicentrality scores of nodes in layer α, or it could be $f(\Phi_{:\alpha})=\|\Phi_{:\alpha}\|_1/n_\alpha$, which denotes the average multicentrality score over nodes in layer α.

There are a variety of ways to define the influence tensor. In this study, we define

$$W^{\alpha}_{\beta}=f(\Phi_{:\alpha})/f(\Phi_{:\beta}). \qquad [2]$$

That is to say, the interlayer influence between 2 layers depends on their relative layer importance. It is clear that there is no gain or loss for links between nodes in the same layer, because the interlayer influence $W^{\alpha}_{\alpha}(\alpha=1,2,\cdots,K)$ is equal to 1. For links from a node in a more important layer, there is a gain in the multicentrality, and vice versa. Then the interaction tensor can be written as $H^{i\alpha}_{j\beta}=\frac{f(\Phi_{:\alpha})}{f(\Phi_{:\beta})}\cdot M^{i\alpha}_{j\beta}$. Further, we prove that $\lambda_\alpha=\lambda_1, \forall\alpha\in\{1,2,\ldots,K\}$, and Eq. **1** reduces to

$$H^{i\alpha}_{j\beta}\Phi_{i\alpha}=\lambda_1\Phi_{j\beta}, \qquad [3]$$

where λ_1 is the leading eigenvalue of the interaction tensor H and is irrelevant to β in this scenario. Notably, λ_1 is also the leading eigenvalue of the adjacency tensor M, indicating that the interaction tensor H maintains the leading eigenvalue of the adjacency tensor M (see *SI Appendix*, section V for further details). From Eq. **3**, we can see how the multicentrality score propagates in a multilayer network. Considering that a node i in layer α links to another node j in layer β, node i will propagate its multicentrality score to node j scaled by an influence coefficient W^{α}_{β}, which could be a gain ($W^{\alpha}_{\beta}>1$), a loss ($W^{\alpha}_{\beta}<1$), or even ($W^{\alpha}_{\beta}=1$), and the multicentrality $\Phi_{j\beta}$ comprises the scores that node j in layer β gathers in the steady state. The existence and uniqueness of the multicentrality Φ are proved in *SI Appendix*, section V. For a given function f, we can calculate the global multicentrality using the compressed power iteration method introduced in *Materials and Methods*.

We consider another interesting scenario in which nodes in different layers are heterogeneous and thus are not comparable. For example, in a heterogeneous network consisting of authors and papers, it is not meaningful to compare the multicentrality of an author with that of a paper. Because the multicentrality of nodes in different layers may have varying implications, we can calculate the local multicentrality of nodes in each layer only while taking into account the interlayer influence, where local multicentrality means that the nodes in each (local) layer are centrality homogeneous. In such a scenario, the multicentrality score of a node cannot simply propagate along interlayer links to other layers. We measure the local multicentrality of nodes in each layer by defining the influence tensor in the framework as

$$W^{\alpha}_{\beta}=\frac{\sum_{i,j=1}^{N}M^{j\beta}_{i\alpha}\Phi_{j\beta}}{\sum_{i,j=1}^{N}M^{i\alpha}_{j\beta}\Phi_{i\alpha}}. \qquad [4]$$

Note that the denominator $\sum_{i,j=1}^{N}M^{i\alpha}_{j\beta}\Phi_{i\alpha}$ in W^{α}_{β} is a normalizing constant, which is the sum of scores propagating along interlayer links from layer α to layer β. Moreover, the sum of interactions from nodes in layer α to nodes in layer β is given by the numerator $\sum_{i,j=1}^{N}M^{j\beta}_{i\alpha}\Phi_{j\beta}$, which is the sum of scores propagating from layer β to layer α. Therefore, by defining the influence tensor in Eq. **4**, we assume that the score flow going out of 1 layer is returned to the layer. In such a way, we can calculate the local multicentrality of nodes in each layer independently while the interlayer influence is taken into account. The detailed proofs of the existence and uniqueness of Φ in local multicentrality are provided in *SI Appendix*, section VI. We can also calculate the local multicentrality Φ numerically using the compressed power iteration method.

Because the prior knowledge is network specific and application dependent, we present 2 interesting scenarios, for global multicentrality and local multicentrality, respectively, to illustrate how to leverage prior knowledge to find W and compute the multicentrality of nodes. The PageRank algorithm has been widely used in social, transportation, biology, and information network analyses for link prediction, recommendation, etc. (32). Note that PageRank centrality is a variant of eigenvector centrality. In PageRank centrality, each node distributes its PageRank score to its neighbors along outgoing links on an equal footing, and a node's PageRank score is defined as the sum of scores that it gathers from its neighbors in the steady state. Our eigenvector multicentrality framework can be easily carried over to characterize PageRank multicentrality (see *SI Appendix*, section IV for further details about PageRank multicentrality).

Multicentrality in Empirical Networks. We first consider a dataset from Wikipedia consisting of 4,604 web pages (see *Materials and Methods* for further details about this dataset and how it has been obtained). The web pages are divided by Wikipedia into 15 subjects, including art, business studies, citizenship, countries, design and technology, everyday life, geography, history, IT, language and literature, mathematics, music, people, religion, and science, and 1 web page belongs solely to 1 subject. We build a multilayer network by placing web pages of the same subject in the same layer and establishing a directed link between 2 web pages if there is a hyperlink between them (a subnetwork is shown in Fig. 1*A*). For comparison, we also build a single-layer network by aggregating all web pages in all layers (see *SI Appendix*, Fig. S4 for further details). We select 4,128 web pages from these, to guarantee that the network is connected and all nodes have at least 1 out-degree and 1 in-degree.

We measure the PageRank multicentrality of web pages in the constructed multilayer network. Moreover, because web pages in different subjects are centrality homogeneous, global PageRank multicentrality is used. We consider 3 common forms for layer importance $f(\cdot)$: $f_1(\Phi_{:\alpha}) = \ln(1 + N \cdot |\Phi_{:\alpha}|_1/n_\alpha)$, $f_2(\Phi_{:\alpha}) = |\Phi_{:\alpha}|_\infty$, and $f_3(\Phi_{:\alpha}) = |\Phi_{:\alpha}|_1/n_\alpha$. Table 1 shows the results of global PageRank multicentrality ($f = f_1$) in the Wikipedia multilayer network and PageRank centrality in the aggregated Wikipedia network, where the digits in parentheses indicate the differences between these 2 rankings (see *SI Appendix*, Tables S1–S3 for further details).

Note that the entry "United States" has the largest multicentrality score, because it has the largest number of incoming links. Furthermore, the web pages linked to it have high multicentrality scores. The entry "Europe" in the layer "Geography" has many enhanced links from nodes in the layer "Countries," because the average PageRank multicentrality score of nodes in Countries is higher than that in Geography. "France" has large numbers of incoming links from entries in the layers "Art," "Music," and other subjects; scores along these incoming links, however, will be diminished since Art and Music are less important than Countries. Note that the entry "Television" is significantly promoted, because the layer importance of "Design and technology" is relatively low. Global multicentrality can effectively quantify the influence between layers even when we have limited prior knowledge, rather than aggregating all nodes without considering interlayer influence, as shown in many previous methods.

Eigenvector multicentrality is an effective predictor for searching for the most important node in multilayer networks, which is also validated by the results obtained from the page views (PVs) in Wikispeedia. Wikispeedia is a human-computation game (33), in which users are requested to navigate from a given web page to a target one by clicking only on Wikipedia links. We collect all completed navigation paths and obtain the PVs of each web page from Wikispeedia. For all entries, we calculate their PageRank multicentrality scores and PageRank centrality and degree centrality scores and compare them to their PVs in Wikispeedia (see *SI Appendix*, Tables S4–S8 for further details). The results are shown in Figs. 2 and 3. We also list the average PageRank multicentrality score of each layer (subjects) in Wikipedia (see *SI Appendix*, Tables S9–S11 for further details). The Spearman rank correlation coefficients show that PageRank multicentrality outperforms PageRank centrality and degree centrality in the aggregated network.

Table 1. Comparison between the ranking by global PageRank multicentrality in the multilayer network and the ranking by PageRank centrality in the aggregated network

Entry	Layer	Global PageRank multicentrality	PageRank centrality
United States	Countries	1	1 (+0)
Europe	Geography	2	3 (+1)
World War II	History	3	7 (+4)
France	Countries	8	2 (−6)
Animal	Science	9	77 (+68)
Christianity	Religion	17	19 (+2)
Earth	Science	18	39 (+21)
20th century	History	26	37 (+11)
Agriculture	Everyday life	28	49 (+21)
Television	Design and technology	61	197 (+136)

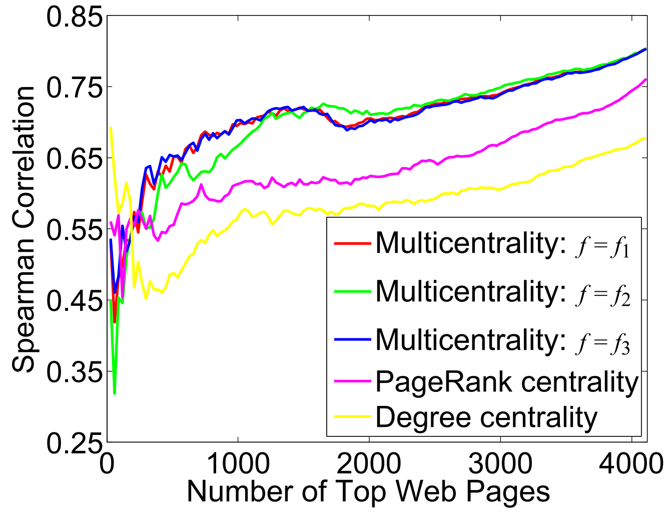

Fig. 2. PageRank multicentrality, PageRank centrality, and degree centrality in the Wikipedia multilayer network. The line chart shows the Spearman correlation coefficients for different numbers of top web pages, where the horizontal axis denotes the number of top nodes that we select, and the vertical axis represents the corresponding Spearman correlation coefficients. When the number of nodes is smaller than 200 (5% of the total nodes), the degree centrality has better performance. However, when more nodes are involved, the multicentrality performs better. In particular, the Spearman correlation coefficients reach 0.80 for all nodes under the 3 proposed multicentrality measures.

Next, we consider a transportation network consisting of airports and air routes between them. We first consider 450 airports in Europe (34) (see *Materials and Methods* for details about this dataset and how it has been obtained), and we focus on 3 main airlines. We then build a multiplex network with 3 layers (airlines) and 450 nodes (airports) in each layer as shown in Fig. 1*B*, where the dotted lines are interlayer links between airports and their counterparts in other layers. We measure the global PageRank multicentrality in the multiplex network using 3 forms of layer importance: $f_1(\Phi_{:\alpha}) = e^{|\Phi_{:\alpha}|_1/n_\alpha} - 1$, $f_2(\Phi_{:\alpha}) = |\Phi_{:\alpha}|_1/n_\alpha$, and $f_3(\Phi_{:\alpha}) = \ln(1 + N \cdot |\Phi_{:\alpha}|_1/n_\alpha)$. Then, the PageRank multicentrality score of each airport is obtained by assembling the multicentrality scores of all its counterparts in all layers. For comparison, we also build a single-layer network, called the aggregated network, by combining the same airports in the 3 layers. We further introduce the versatility, a good predictor for diffusive and congestion processes in multilayer networks (23), which is a special case of our framework when setting all components of the influence tensor W to 1. We focus on the coverage $\rho(t)$, a suitable proxy for the exploration efficiency of the network (35), defined as the average fraction of distinct nodes being visited up to time t regardless of the

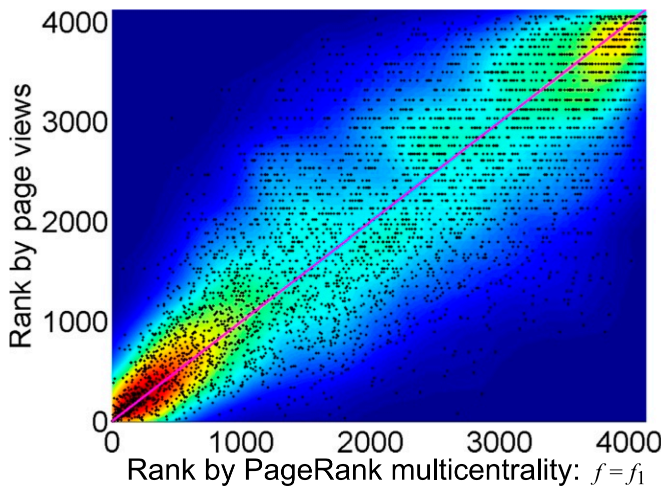

Fig. 3. Scatter diagram of the ranks of the entries by multicentrality and by PVs. The diagram shows the PageRank multicentrality of all nodes when $f = f_1$ (see *SI Appendix*, Fig. S5 for additional diagrams), where the horizontal axis denotes the rank of the entries by multicentrality and the vertical axis represents the rank of the entries by PVs in Wikispeedia. The Spearman correlation coefficient between the 2 rankings is 0.80.

layer, assuming that a walker starts from a certain node in the network (see *Materials and Methods* for further details about the coverage). We investigate whether multicentrality helps to understand the role that a node plays in dynamical scenarios. To this end, we compute the Spearman correlation coefficient between the ranking of the airports by multicentrality and that by the coverage at time t of a hypothetical epidemic spreading process that starts from a certain airport. For comparison, we also compute baselines such as the rankings by versatility and PageRank centrality in the aggregated network (see *SI Appendix*, Tables S12–S14 for further details). We calculate the Spearman correlation coefficients for these 5 methods at each time step, and the results are shown in Fig. 4, where the time ranges from $t = 1$ to $t = 4,000$. It is shown that the 3 multicentrality measures achieve higher accuracy (their correlation coefficients exceed 0.947) in the steady state ($t \geq 3,000$). We then perform a similar analysis on an airline network from the United States, which contains the airlines flying from the United States on Jan 3, 2008 (with data provided by the American Statistical Association Sections on Statistical Computing, http://stat-computing.org/). We build a multiplex network with 20 layers and 284 nodes in each layer, and a similar conclusion can be drawn (see *SI Appendix*, Table S15 for further details).

Another real-world example we consider is a social network, constructed from a large European research institution with 1,005 nodes (individuals) and 42 layers (departments), on which we consider an epidemic spreading process. The simulation results indicate that the nodes with higher eigenvector multicentrality play a more important role in the epidemic spreading process (see *SI Appendix*, section VII for details).

Discussion

As shown in recent work on eigenvector centrality (and its variants) (36–38), it is of significant interest to build a framework for studying eigenvector-like centrality in multilayer networks. The existing studies, however, assumed empirical influence coefficients or relied on specific types of multilayer networks. Here we develop a general framework for studying eigenvector multicentrality in multilayer networks, which enables the quantification of the impact of interlayer influence on eigenvector multicentrality, providing an analytical tool to describe how eigenvector multicentrality propagates among different layers. Further, this framework can easily leverage prior knowledge about the interplay among layers to characterize eigenvector multicentrality for varying scenarios. As the interlayer influence and multicentrality of nodes are interdependent, they are jointly solved using a compressed power iteration method. Furthermore, we formulate and analyze PageRank multicentrality for practical applications within the proposed framework. We also perform theoretical analyses to prove the existence and uniqueness of the solutions in *SI Appendix*, sections V and VI, which allows us to calculate global and local multicentrality in any strongly connected multilayer networks. For an arbitrary multilayer network, we treat a dead end (a node with no outgoing links) the same as if it had outgoing links to all nodes and introduce a damping factor to guarantee the existence and uniqueness of the solution. The results from empirical networks demonstrate that our general framework can effectively quantify the interlayer influence, and eigenvector multicentrality is a good measure to identify important nodes from both structural and dynamical perspectives. Thus, multicentrality aids in understanding and predicting the behaviors of dynamic processes by leveraging network structure and describes the structure–function relationship of multilayer networks well.

We believe that the concept of multicentrality has the potential to offer a deep understanding of the structure and function of multilayer networks. Because the real-world scenarios of multilayer networks vary, a key step is to find appropriate forms of interlayer influence for theoretical analyses. Here we consider 2 interesting scenarios, for global multicentrality and local multicentrality, respectively. We believe that the proposed tensor-based framework can be applied to more empirical networks in various scenarios, including social networks, transportation networks, biological networks, etc.

Materials and Methods

Numerical Solution. The crux of the proposed framework is to solve the tensorial equation

$$H^{i\alpha}_{j\beta}\Phi_{i\alpha} = \lambda_{\beta}\Phi_{j\beta}, \quad [5]$$

where the solution Φ is the multicentrality tensor. To obtain the numerical solution, we first flatten the adjacency tensor M into a matrix; i.e., we represent the fourth-order tensor $M \in \mathbb{R}^{N\times N\times K\times K}$ as a matrix $\overline{M} \in \mathbb{R}^{NK\times NK}$, where $\overline{M}$ denotes the lower-dimensional form of the tensor M. Then we

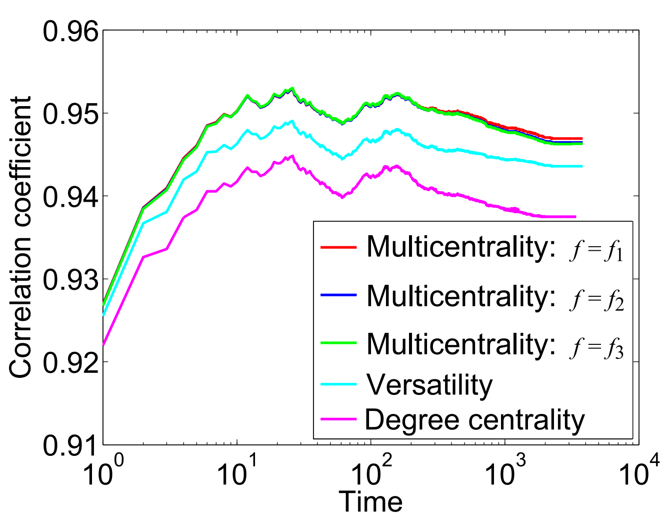

Fig. 4. Comparison of PageRank multicentrality, PageRank versatility, and degree centrality for the European airline network. The Spearman correlation coefficients in the dynamical process are plotted. When $t \geq 3,000$, these 5 curves of correlation coefficients tend to be stable, and there is a gap between the multicentrality measure and other measures.

vectorize the second-order tensor $\Phi \in \mathbb{R}^{N\times K}$ into a supravector $\overline{\Phi} \in \mathbb{R}^{NK}$ and denote by $\overline{W}$ the matrix form of the influence tensor W (see *SI Appendix*, section II for further details of tensor decomposition). Further, we denote by $\Lambda = [\lambda_1, \lambda_2, \cdots, \lambda_K]^\top \in \mathbb{R}^K$ a vector encoding the normalizing coefficient in each layer. Thus, we obtain the matrix equation

$$\left(\overline{W} \odot \overline{M}\right)^\top \cdot \overline{\Phi} = \Lambda \odot \overline{\Phi}, \quad [6]$$

where $\odot$ denotes the Khatri–Rao product and $\overline{W} = \overline{W}(\overline{M}, \overline{\Phi})$ is a function of the matrix $\overline{M}$ and the multicentrality $\overline{\Phi}$.

For the numerical solution, we propose a compressed power iteration method, whose iteration scheme is

$$\overline{\Phi}^{(k+1)} = \overline{\Phi}^{(k)} + D^{(k)} \left[\left(\overline{W} \odot \overline{M}\right)^{\top(k)} \odot \Omega^{(k)} - E_N\right]\overline{\Phi}^{(k)}, \quad [7]$$

where $\Omega^{(k)} = [\omega_1^{(k)}, \omega_2^{(k)}, \cdots, \omega_K^{(k)}] \in \mathbb{R}^K$ is the normalizing vector. Denoting $B^{(k)} = (\overline{W} \odot \overline{M})^{\top(k)} \odot \Omega^{(k)} - E_N$, we can write the iteration scheme as

$$\overline{\Phi}^{(k+1)} = \overline{\Phi}^{(k)} + D^{(k)} B^{(k)} \overline{\Phi}^{(k)}, \quad [8]$$

where $D^{(k)} \in \mathbb{R}^{N\times N}$ is a diagonal matrix related to $B^{(k)}$, and $D^{(k)}$ compresses the induced infinity norm of the matrix $B^{(k)}$ such that the L^1 norm of each row in $D^{(k)}B^{(k)}$ is strictly less than 1.

Specifically, we let $\omega_\gamma^{(k)} = \|\overline{\Phi}_\gamma^{(k)}\|_1^{-1}$. With regard to the global multicentrality, the vector Λ contains equivalent elements; i.e., $\Lambda = [\lambda_1, \lambda_1, \cdots, \lambda_1]^\top$. Thus, we have $\Omega^{(k)} = [\omega_1^{(k)}, \omega_1^{(k)}, \cdots, \omega_1^{(k)}]^\top$, where $\omega_1^{(k)} = \|\overline{\Phi}^{(k)}\|_1^{-1}$. Further, we can specify the diagonal matrix $D^{(k)}$ to compress the induced infinity norm of matrix $B^{(k)}$, where $D^{(k)}$ is not unique in practice. For example, we could take $D^{(k)} = \left(\text{diag}\left\{(B^{(k)} + E_N) \cdot \mathbf{1}_{N\times 1}\right\}\right)^{-1}$. Then for each $\overline{\Phi}^{(k)} > 0$ (i.e., all of the components in $\overline{\Phi}^{(k)}$ are positive), the matrix $[E_N + D^{(k)}B^{(k)}]$ is strictly diagonal dominant with positive elements. Hence, the iterations in Eq. **8** converge to a unique solution (39, 40) and this solution satisfies the matrix Eq. **6** (see *SI Appendix*, section VIII for further details).

Multilayer Network of Wikipedia. The Wikipedia dataset contains 4,604 entries and 119,882 hyperlinks (41) (data provided by the Stanford Network Analysis Project, http://snap.stanford.edu/index.html). To ensure connectivity, we select the entries that have at least 1 out-degree and 1 in-degree. Then 4,128 entries and 113,441 links are obtained and these entries are divided into 15 subjects by Wikipedia. Resultantly, we can build a multilayer network with $N = 4,128$ nodes and $K = 15$ layers, where each layer contains the entries of a single subject. The adjacency tensor $M \in \mathbb{R}^{N\times N\times K\times K}$ encodes the directed and unweighted links of the multilayer network and the influence tensor $W \in \mathbb{R}^{K\times K}$ encodes the interlayer influence between any 2 layers. We measure the interlayer influence

$$W_\beta^\alpha = f(\Phi_{:\alpha})/f(\Phi_{:\beta}), \quad [9]$$

where $f(\Phi_{:\alpha})$ indicates the layer importance of layer α. Here, we consider 3 forms of layer importance: $f_1(\Phi_{:\alpha}) = \ln(1 + N \cdot |\Phi_{:\alpha}|_1/n_\alpha)$, $f_2(\Phi_{:\alpha}) = |\Phi_{:\alpha}|_\infty$, and $f_3(\Phi_{:\alpha}) = |\Phi_{:\alpha}|_1/n_\alpha$. Further, we obtain the interaction tensor $H \in \mathbb{R}^{N\times N\times K\times K}$ as $H_{j\beta}^{i\alpha} = W_\beta^\alpha M_{j\beta}^{i\alpha}$. Following the construction of the interaction tensor H, we then solve the tensorial equation

$$H_{j\beta}^{i\alpha} \Phi_{i\alpha} = \lambda_1 \Phi_{j\beta} \quad [10]$$

using the compressed iteration method. Finally, the multicentrality tensor Φ is in the space $\mathbb{R}^{N\times K}$, and $\Phi_{i\alpha}$ represents the multicentrality of node i in layer α.

Multilayer Network of the European Airlines. The European airline network contains 450 airports in Europe and the air routes for 37 airlines (see ref. 34 for more details about this dataset). For each airline, we can build a network with $N = 450$ nodes and a set of links representing routes between airports. We select those airlines with the number of air routes greater than $N/2$, such that the average degree for each node in the constructed network is at least 1. In this way, we obtain 3 main airlines: Ryanair, Lufthansa, and Easyjet. We thus have a multiplex network with $K = 3$ layers and 450 nodes in each layer. Then, we interconnect the same airport across layers, obtaining a 3-layer multiplex network. The adjacency tensor M is in the space $\mathbb{R}^{N\times N\times K\times K}$. In the context of the multiplex network, $M_{j\alpha}^{i\alpha}$ encodes the undirected and unweighted intralayer links in layer α, while $M_{j\beta}^{j\alpha} = 1$ encodes the undirected and unweighted interlayer link for node j between layer α and layer β. For the interlayer influence, we again consider 3 forms of the layer importance: $f_1(\Phi_{:\alpha}) = e^{|\Phi_{:\alpha}|_1/n_\alpha} - 1$, $f_2(\Phi_{:\alpha}) = |\Phi_{:\alpha}|_1/n_\alpha$, and $f_3(\Phi_{:\alpha}) = \ln(1 + N \cdot |\Phi_{:\alpha}|_1/n_\alpha)$. After obtaining the influence tensor W via Eq. **2**, we have the interaction tensor $H_{j\beta}^{i\alpha} = W_\beta^\alpha M_{j\beta}^{i\alpha}$ in the space $\mathbb{R}^{N\times N\times K\times K}$. Finally, we solve the tensorial Eq. **3** using the compressed iteration method and obtain the multicentrality tensor $\Phi_{i\alpha}$.

With respect to the coverage $\rho(t)$, we have

$$\rho(t) = 1 - \frac{1}{N^2} \sum_{i,j=1}^{N} \delta_{i,j}(0) exp\left[-\mathbf{P}_j(0)\mathbb{P}\mathbf{E}_i^\top\right], \quad [11]$$

where $\delta_{i,j}(0) = 0$ for $j = i$, and $\delta_{i,j}(0) = 1$ otherwise. Here, $\mathbf{P}_j(0)$ represents the supravector of probabilities at time $t = 0$ (assuming that the walker starts at node j) and the matrix $\mathbb{P}$ indicates the probability of reaching each node through any path of length $1, 2, \ldots, t + 1$. Furthermore, $\mathbf{E}_i = (\mathbf{e}_i, \mathbf{e}_i, \ldots, \mathbf{e}_i)$ is the supravector in which $\mathbf{e}_i$ is the ith canonical row vector (see ref. 35 for details about the derivation of Eq. **11**).

ACKNOWLEDGMENTS. We thank Manlio De Domenico for sharing the processed dataset of the European airline network and the details for calculating the versatility. This work was supported by the National Natural Science Foundation of China (Grant 61731004) and the Zhejiang Natural Science Foundation (Grant LR16F020001). This work was supported in part by the US Army Research Office under Grant W911NF-16-1-0448 and Defense Threat Reduction Agency under Grant HDTRA1-13-1-0029.

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